Net Present Value

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LKCSB 5036

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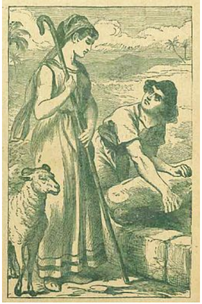
A Love Story, in 1575 BC

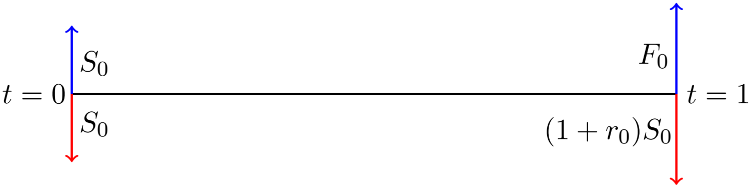
- Jacob met Rachel at the well.
- Jacob entered into a forward contract with Laban, Rachel's father.
  - Buyer Jacob
  - Seller: Laban
  - Underlying asset: Rachel
  - Maturity: 7 Years
  - Settlement: Physical delivery at maturity
  - Forward price of asset: Equivalent of 7 years' slavish labor
- First ever forward contract?
- First ever counterparty default!



Picture source: Gutenberg

Fair Forward Price

- $t = 0$: time of forward contract initiation
- S_0: underlying asset's price at time 0
- r_0: risk-free interest rate at time 0
- F_0: the fair forward price of the forward contract
- $t = 1$: A year later, the forward contract matures.



The Cash Flows of Forward Seller

Self-Financing

- At time 0
  - No cash flow at the initiation of a forward contract
  - Borrow the amount S_0 at the risk-free rate of r_0
  - Buy the underlying at the price of S_0
  - Net cash flow or net present value of the contract is $S_0 - S_0 = 0$.

- Since the net cash flow is zero, the short position in the forward contract is said to be self-financing.

- At time 1 (year)
  - Sell the asset for F_0 to the forward buyer
  - Return the principal plus interest $(1 + r_0)S_0$
  - Net cash flow = $F_0 - S_0(1 + r_0)$

Application of Third Principle

- If the net cash flow at time 1 is positive, i.e., $F_0 > S_0(1 + r_0)$, the forward buyer won't be happy and so won't trade because F_0 is too high.
- Conversely, if $F_0 < S_0(1 + r_0)$, seller is losing money because F_0 is too low and so won't trade.
- Since S_0, r_0, and F_0 are known and to be determined at time 0, the only way both the buyer and the seller are happy to trade is to have

$$F_0 = S_0(1 + r_0) \tag{1}$$

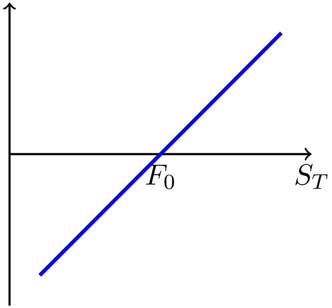
- Otherwise, no trade will occur at time 0.
- Simply, F_0 is the forward value of S_0.

Discussion

- Suppose short selling is permitted, and the proceeds can be fully utilize to invest in risk-free security.

- From the forward buyer's point of view, what is the self-financing strategy for determining F_0?

Linear Payoff



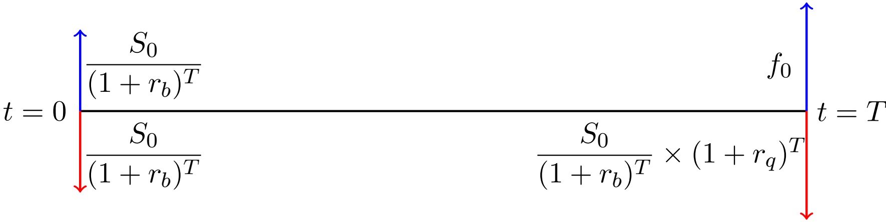
The payoff of the forward buyer at maturity T.

- The buyer is obligated to buy the asset at F_0.
- Compare against the spot price S_T of the underlying asset at maturity time T, the forward buyer's payoff (P&L on paper) is linear:

$$S_T - F_0$$

Cash Flows of Forward FX Contract

- S_0: spot FX rate in base currency/quote currency
- f_0: forward FX rate in base currency/quote currency
- r_b: risk-free rate for fixed income security in base currencies
- r_q: risk-free rate for fixed income security in quote currencies
- T: time to maturity



The Cash Flows (in Quote Currencies) of Forward FX Seller

Interest Rate Parity

- For the trade to be possible, by the third principle of $\mathbb{Q}$F, it must be that

$$\boxed{f_0 = \frac{(1+r_q)^T}{(1+r_b)^T} S_0.} \tag{2}$$

- Indeed, r_q is the earlier risk-free rate r_0, for stocks are transacted in the quote currency.
- In other words, the forward exchange rate f_0 can be written as

$$f_0 = \frac{F_0}{(1+r_b)^T},$$

where F_0 is expressed in (1) with $r_0 = r_q$.

- The novelty here is the "discount factor" $\dfrac{1}{(1+r_b)^T}$. Why?

Forward FX Rates in Practice

- In practice, the rate for a forward FX deal is generally expressed as the amount by which the forward rate diverges from the spot rate.

$$f_0 - S_0 = \frac{(1+r_q)^T - (1+r_b)^T}{(1+r_b)^T} S_0.$$

  This difference is called the forward margin, also known as the swap point.

- If the swap point is negative, the base (foreign) currency is said to be trading at a forward discount to the quote (domestic) currency.

Interest Rate Spread

- As a percentage per annum, we write

$$\frac{f_0 - S_0}{S_0} = \frac{(1+r_q)^T - (1+r_b)^T}{(1+r_b)^T} \approx (r_q - r_b)T.$$

- The forward FX deal is really a trade on the difference or the spread between the two interest rates r_b and r_q of tenor T. These two rates are the yields of debt securities issued by the governments of the base and quote currencies, respectively.

- So now you know everyone in the FX market is watching what the central banks are going to do to their target interest rates.

Non-Deliverable Forward

- Thus far, we have assumed that the forward contract binds the two counterparties to a physical exchange of funds at maturity.
- By contrast, non-deliverable forward (NDF) is an outright forward contract in which counterparties settle the difference between the contracted forward rate and the prevailing spot price rate on an agreed notional amount.
- NDF-implied yield on the capital-controlled currency offshore

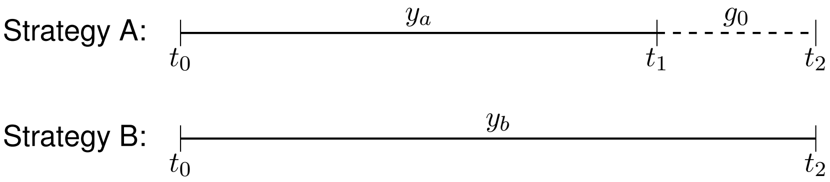
$$f_0^* = \frac{(1+r_i)^T}{(1+r_b)^T} S_0.$$

Introduction

- Derivatives of interest rates are ubiquitous and crucially important in managing interest rate risks, banks' asset and liability.
- Main products are forward rate agreements (FRAs), interest rate swaps (IRS), and interest rate options
- According to BIS' 2013 Triennial Central Bank Survey statistic, the OTC interest rate derivatives turnover was 2.343 trillion US dollars *per day* on average.

Forward Interest Rate

- y_a: risk-free yield of tenor $t_1 - t_0$
- y_b: risk-free yield of tenor $t_2 - t_0$
- g_0: (implied) forward interest rate



Two Strategies that Give Rise to the Same Forward Value

Forward Interest Rate (Cont'd)

- By the first and third principles of $\mathbb{Q}$F,

$$(1 + y_a)^{t_1 - t_0} \times (1 + f_0)^{t_2 - t_1} = (1 + y_b)^{t_2 - t_0} \quad (3)$$

- Solving for f_0, we obtain

$$f_0 = \left(\frac{(1 + y_b)^{T_2}}{(1 + y_a)^{T_1}} \right)^{\frac{1}{T_2 - T_1}} - 1.$$

For notational convenience, we have let $T_1 := t_1 - t_0$ and $T_2 := t_2 - t_0$.

Forward Rate Agreement

- In a typical FRA, one of the counterparties (A) agrees to pay the other counterparty (B) LIBOR settling t years from now applied to a certain notional amount (say, $500 million).

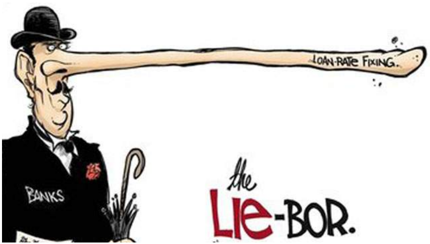
- In return, counterparty B pays counterparty A a pre-agreed interest rate (say, 1.05%) applied to the same notional.

- The contract matures on day T (say, 3 months) from the settlement date, and interest is computed on an actual/360 day count basis.

ICE LIBOR

- LIBOR: London Interbank Borrowing Offer Rate
- Survey question for daily fixings by Intercontinental Exchange (ICE)
  "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am London time?"
- The highest 25% percent responses and lowest 25% responses are eliminated from the data set and the remaining responses are averaged. The average of the rates equals LIBOR for the particular currency and duration.
- Is it possible to move LIBOR either up or down by a submission intended to manipulate?

Ethics: BBA LIBOR Scandal



Source:

http://heavyeditorial.files.wordpress.com/2012/12/libor-111.jpg

NEVER succumb to "collaboration" in the grey area!

- "Hi Guys, We got a big position in 3m libor for the next 3 days. Can we please keep the lib or fixing at 5.39 for the next few days. It would really help. We do not want it to fix any higher than that. Tks a lot."

  – Senior trader in New York to submitter

- Check out what's behind the Libor Scandal.

Fair FRA Rate K

- Two counterparties that have entered into an FRA are obligated to exchange cash flow in the future based on a predetermined strike rate K and a forward spot rate R, which becomes observable at forward time.

- In practice, the strike rate K is referred to as the FRA rate, and the future spot rate R as the fixing rate.

- There is no cash flow at the current time t_0 when the FRA is dealt. The counterparties, among other things, agree upon the strike rate K that is "fair" to both parties.

FRAs of Short-Term Maturities

➦ The fair value K is given by the following relationship:

$$(1+\tau_1 r_1)(1+\tau_k K) = 1 + (\tau_1 + \tau_k) r_2, \tag{4}$$

where

- r_1 is the spot rate with a shorter maturity τ_1.
- τ_k is the FRA maturity
- r_2 is the spot rate with maturity $\tau_1 + \tau_k$.

➦ It follows from (4) that the FRA rate is given by

$$\boxed{K = \frac{1}{\tau_k}\left(\frac{1+(\tau_1+\tau_k)r_2}{1+\tau_1 r_1} - 1\right).} \tag{5}$$

Discount Factor

- The discount factor is a quantity used for discounting the future cash flow as a function of time to maturity and an interest rate.
- Each future cash flow C_i $(i = 1, 2, \ldots, n)$ is receivable at time τ_i with respect to today (time 0).
- The present value for the stream of cash flows is then obtained as follows:

$$\text{PV} = \sum_{i=1}^{n} \text{DF}_i \times C_i.$$

Discount Factor (Cont'd)

- Given the yield curve of zero-coupon bonds with rate z_i, for Treasury bond paying coupons semi-annually, we have

$$\boxed{\mathsf{DF}_i = \frac{1}{\left(1+\frac{z_i}{2}\right)^i},} \tag{6}$$

- The compounding scheme of (4) is, as anticipated,

$$\mathsf{DF}_\tau = \frac{1}{1+\tau r}. \tag{7}$$

- Corresponding to the two short-term maturities τ_1 and $\tau_1 + \tau_k$, the discount factors are, respectively,

$$\mathsf{DF}_1 = \frac{1}{1+\tau_1 r_1} \quad \text{and} \quad \mathsf{DF}_k = \frac{1}{1+(\tau_1+\tau_k) r_2}.$$

Tutorial

1. Show that the FRA rate (5) can be written as a function of discount factors:

$$K = \frac{1}{\tau_k}\left(\frac{\mathsf{DF}_1}{\mathsf{DF}_k} - 1\right). \tag{8}$$

2. A U.S. Treasury bond has one year remaining to maturity. Express the annual coupon rate c in terms of the yield y to maturity, and the discount factors in the form of (6).
Hint:

$$\mathsf{PV} = \frac{\frac{c}{2}}{1+\frac{y}{2}} + \frac{1+\frac{c}{2}}{\left(1+\frac{y}{2}\right)^2} = \frac{c}{2}\mathsf{DF}_1 + \left(1+\frac{c}{2}\right)\mathsf{DF}_2$$

FRA's Payoff is Linear

- At time τ_1 when the FRA expires, the LIBOR rate R of tenor τ_k is observed. The cash flow to the buyer is then given by

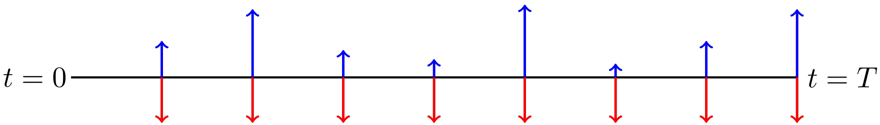
$$\text{Notional Amount} \times (R - K)\tau_k \left(\frac{1}{1 + R\tau_k} \right).$$

- The cash flow generated by the interest rate differential is discounted by the discount factor $\dfrac{1}{1 + R\tau_k}$.

- This is because instead of entering into the "physical" or actual borrowing over the tenor of τ_k starting from τ_1, the anticipated cash flow at $\tau_1 + \tau_k$, namely, notional Amount $\times (R - K)\tau_k$, is settled at τ_1 by discounting it back from $\tau_1 + \tau_k$ to τ_1.

Definition of Interest Rate Swap

- According to the definition by ISDA, interest rate swap (IRS) is an agreement to exchange interest rate cash flows, calculated on a notional principal amount, at specified intervals (payment dates) during the life of the agreement.
- Each party's payment obligation is computed using a different interest rate.



The cash flows of interest rate swap buyer over 8 quarters since deal date.

Fixed Leg of the IRS

→ A bond selling at par with n coupons at a fixed coupon rate of c per period.

$$1 = c \sum_{i=1}^{n} \mathrm{DF}_i + \mathrm{DF}_n \times 1. \tag{9}$$

→ The fixed rate K for the fixed leg of the IRS is determined as if a bond is issued at par value of 1 with $c = K$:

$$1 = K \sum_{i=1}^{n} \mathsf{DF}_i + \mathsf{DF}_n \,. \tag{10}$$

Net Present Value

- The net present value of the IRS at time 0 is

$$\text{NPV}_0 = \left(\sum_{j=1}^{n} \text{DF}_j \times \text{Floating CF}_j + \text{DF}_n \times 1 \right) - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).$$

- In this form, IRS is effectively a long-short strategy on two bonds. The IRS buyer is effectively betting on a position that is long in the floating rate security and short in the fixed rate bond.

Net Present Value (Cont'd)

→ At time 0, since both bonds are issued at par, by the third law of $\mathbb{Q}$F, we must have $\text{NPV}_0 = 0$. Accordingly, we set the floating bond to its par value to obtain

$$0 = 1 - \sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i - \text{DF}_n \times 1.$$

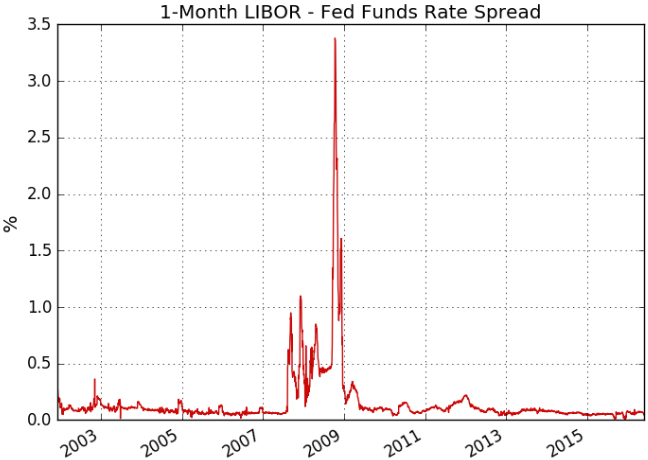
→ Result: Pricing the IRS

$$\boxed{K = \frac{1 - \text{DF}_n}{\sum_{i=1}^{n} \text{DF}_i}.} \tag{11}$$

Overnight Index Swaps (OIS)

- Overnight indexed swaps are interest rate swaps in which a fixed rate of interest (OIS rate) is exchanged for a floating rate that is the geometric mean of a daily overnight rate.
- The overnight rates include
  - Federal Funds rate (USD)
  - EONIA (EUR)
  - SONIA (GBP)
  - CHOIS (CHF)
  - TONAR (JPY)
- There has recently been a shift away from LIBOR-based swaps to OIS indexed swaps due to the scandal.
- Discounting with OIS is now the standard practice for pricing collateralized deals and is being mandated by clearing houses.

LIBOR-OIS Spread



The spread became most noticeable during the credit crisis.

LIBOR-OIS Spread (Cont'd)

"Libor-OIS remains a barometer of fears of bank insolvency."

Source: "What the Libor-OIS Spread Says," Economic Synopses 2009, Number 24

Alan Greenspan

"I made a mistake in presuming that the self-interests of organizations, specifically banks and others, were such as that they were best capable of protecting their own shareholders and their equity in the firms"

Source: The New York Times, Oct 23, 2006

Conceptual Check: Which is the Odd One out?

1. The Fed Funds rate is determined by the supply and demand in the interbank lending and borrowing market.
2. The LIBOR − OIS is the gain to an interest rate swap buyer.
3. Interest rate swap buyer is disadvantaged because his cash flow is uncertain.
4. OIS rate is the fixed rate in an interest rate swap.

All Kinds of Curves

- From zero rates, you obtain a curve of discount factors (discount curve)

$$\text{DF}_j = \frac{1}{\left(1 + \frac{z_j}{2}\right)^j}$$

- From zero rates, you obtain the forward interest rates, and plot them against their respective maturities.

- From zero rates, you can compute the par rates c. For example

$$\frac{\frac{c}{2}}{1 + \frac{z_1}{2}} + \frac{\frac{c}{2} + 100}{\left(1 + \frac{z_2}{2}\right)^2} = 100.$$

Forward-Forward Rates

- Let $f(t-1,t)$ be the annualized implied forward (forward-forward) for lending/borrowing start at time $t-1$ till t.
- The bond price can also be written as

$$
\begin{aligned}
P &= \frac{\frac{C}{2}}{1+\frac{f(0,1)}{2}} + \frac{\frac{C}{2}}{\left(1+\frac{f(0,1)}{2}\right)\left(1+\frac{f(1,2)}{2}\right)} + \cdots \\
&\quad \cdots + \frac{A}{\left(1+\frac{f(0,1)}{2}\right)\cdots\left(1+\frac{f(T-1,T)}{2}\right)} \\
&= \frac{C}{2}\sum_{t=0}^{T}\frac{1}{\prod_{i=1}^{t}\left(1+\frac{f(i-1,i)}{2}\right)} + \frac{A}{\prod_{i=1}^{T}\left(1+\frac{f(i-1,i)}{2}\right)}
\end{aligned}
$$

Forward-Forward Rates (Cont'd)

- The spot zero rate is essentially the geometric average of the forward rates

$$\left(1+\frac{z}{2}\right)^t = \left(1+\frac{f(0,1)}{2}\right)\left(1+\frac{f(1,2)}{2}\right)\cdots\left(1+\frac{f(t-1,t)}{2}\right)$$

- The implicit relationship between the spot and forward interest rates is

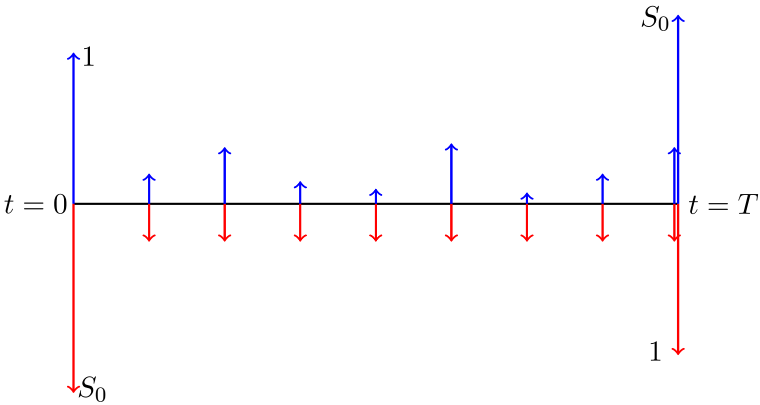
$$1+\frac{f(t-1,t)}{2} = \frac{\left(1+\frac{z_t}{2}\right)^t}{\left(1+\frac{z_{t-1}}{2}\right)^{t-1}} = \frac{\mathsf{DF}_{t-1}}{\mathsf{DF}_t}.$$

Multi-Curve Approach to Price Interest Rate Swaps

- Before the 2008 financial crisis, the discount curve and the forward curve are based on LIBOR. You just need to construct the LIBOR forward curve to obtain the swap rates.
- After the crisis, a common practice is to use the multi-curve approach based on OIS discounting. The discount factors are computed from OIS rates instead.
- Moreover, for the floating leg, you need to build separate 1-month, 3-month LIBOR forward curves to account for the tenor.

Cash Flows of CIRS

- The cross-currency interest rate swap (CIRS) may be regarded as a generalized version of an IRS.



The cash flows of cross-currency interest rate swap buyer over 8 quarters since deal date. S_0 is the FX rate of the quote currency.

NPV Pricing of CIRS

- Given the spot FX rate S_0, which is the units of quote currency needed to exchange for one unit of base current, the net present value for the CIRS buyer is

$$\begin{aligned}\text{NPV}_0 = & S_0 \left(\sum_{j=1}^{n} \text{DF}_j \times \text{Floating CF}_j + \text{DF}_n \times 1 \right) \\ & - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).\end{aligned}$$

- The buyer receives the base currency in exchange for the quote currency at the spot rate S_0.

NPV Pricing of CIRS (Cont'd)

- Again, this is a long-short strategy. The CIRS buyer is long a floating bond denominated in the base currency and short in a fixed rate bond in the quote currency.
- What is the value of NPV_0 at time 0?

  Answer: ________________________________
- Floating leg's bond is valued at par.

$$S_0 - 1 = S_0 - \left(\sum_{i=1}^{n} \text{DF}_i \times \text{Fixed CF}_i + \text{DF}_n \times 1 \right).$$

- Solving for K, we find that the fixed rate is still given by the same formula: (11)!

Takeaways

- Pricing of plain vanilla forward and FX forward by the three principles of $\mathbb{Q}$F.
- Key concept: Self-financing strategy
- Pricing of forward rate agreement, interest rate swap, and cross-currency interest rate swap by the three principles of $\mathbb{Q}$F.
- All these derivatives have linear payoffs.
- Many different curves are needed for pricing interest rate derivatives.

Week 5 Assignment from Chapter 5

Question 1

Question 1 of textbook's Chapter 5

Question 2

Starting from the result in Problem 2 of the tutorial in Slide 24, show that

$$\frac{1}{1+\frac{y}{2}} + \frac{2}{\left(1+\frac{y}{2}\right)^2} > \mathrm{DF}_1 + 2\mathrm{DF}_2.$$

Week 5 Additional Exercises

1. Question 2 of Chapter 5

2. Show that the following relationship holds in the real world for a pair of currencies that has 1-month forward exchange rate F_{1m} and 3-month forward exchange rate F_{3m}:

$$\frac{90F_{1m} - 30F_{3m}}{S} \approx 60.$$

The spot rate is denoted by S.